

A comment on the paper "Deformed Boost Transformations that saturate at the Planck Scale"

by N.B.Bruno,G.Amelino-Camelia, and
J.Kowalski-Glikman

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Abstract

An alternative (simplified) derivation of the dispersion relation and the expressions for the momentum-energy 4-vector p_i, p_0 given initially in [1] is provided. It has turned out that in a rather "pedestrian" manner one can obtain in one stroke not only the above relations but also the correct dispersion relation in $\omega - k_i$ space, consistent with the value of a velocity of a massless particle. This is achieved by considering the standard Lorentz algebra for $\omega - k_i$ -space. A non-uniqueness of the choice of the time-derivative in the presence of the finite length scale is discussed. It is shown that such non-uniqueness does not affect the dispersion relation in $\omega - k_i$ -space, albeit results in different dispersion relations in $p - p_0$ -space depending on the choice of the definition of the time derivative.

1 Introduction

In this paper we revisit the work by N.B.Bruno,G.Amelino-Camelia, and J.Kowalski-Glikman ^[1] on finite boost transformations (BAK transformations) following from the κ -Poincare algebra. The prime motivation of our return to these results was that in the space of the boost parameter ξ the commutation relations of the phase portion of the algebra are pronouncedly non-symmetric.

As a result this leads to somewhat involved exact explicit expressions for the spatio-temporal components of the four-momentum as functions of the

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boost parameter ξ . We *re-derive* all their results by introducing Lorentz algebra (a subalgebra of the original algebra). Here the original components of the four-momentum [1] are replaced by the transformed quantities. The resulting equations are amazingly simple (in this sense our work follows the approach of [1] where the difficulties encountered in an earlier work [2] have been obviated due to a judicious choice of the appropriate κ -Poincare algebra) and allow one to find a very simple form of the explicit expressions obtained in [1] for the original quantities as functions of ξ , the dispersion equation for the old and transformed quantities. The latter turned out to be an exact analog of the respective relation in special relativity. The kinematic part of the respective Lorentz algebra requires an introduction of the transformed boost parameter $z = z(\xi)$ [3]. This tells us that phase and position sector of the κ -Poincare algebra cannot be "diagonalized" simultaneously.

2 Finite boost transformations in standard Lorentz algebra and their relation to the BAK transformations

Our point of departure is the commutation relations [1] of κ -Poincare algebra pertinent to the finite boost transformations :

$$\begin{aligned} [N_j, p_r] &= i\delta_{jr} \left(\frac{1 - e^{-p_0\lambda}}{\lambda} + \frac{\lambda}{2} |\vec{p}|^2 \right) - i\lambda p_j p_r, \\ [N_j, p_0] &= ip_j \end{aligned} \quad (1)$$

where p_0, p_j are the temporal and spatial components of the four-momentum, λ is the observer-independent scale, $j, r = 1, 2, 3$, and N_j are the boost generators.

We introduce new transformed quantities n_j, ω, k_j (where the latter represent the frequency-wave number four-vector) such that the respective commutation relations are standard Lorentzian:

$$\begin{aligned} [n_j, k_r] &= i\delta_{jr}\omega \\ [n_j, \omega] &= ik_j \end{aligned} \quad (2)$$

In the following we restrict our attention to the boost in the direction of the axis 1. Clearly this does not lead to a loss of generality since we can always choose the direction of boost as the axis 1. The differential equations

corresponding to the differential representation of n_1 are

$$\begin{aligned}\frac{d\omega}{d\xi} &= k_1 \\ \frac{dk_1}{d\omega} &= \omega\end{aligned}\tag{3}$$

For our purposes we rewrite commutation relations (2) in a slightly different form:

$$\begin{aligned}[n_1, |k|^2] &= i \frac{\omega}{|k|} k_1 \\ [n_1, \omega] &= i k_1\end{aligned}\tag{4}$$

Therefore the differential representation of the new boost generator n_1 is

$$n_1 = i \frac{\omega}{|k|} k_1 \frac{\partial}{\partial |k|} + i k_1 \frac{\partial}{\partial \omega}\tag{5}$$

The respective characteristic equations to be satisfied by (ω, \vec{k})

$$\begin{aligned}\frac{d\omega}{d\xi} &= k_1 \\ \frac{d|k|}{d\xi} &= \frac{\omega}{|k|} k_1\end{aligned}\tag{6}$$

yield the familiar dispersion equation of the special relativity in (ω, \vec{k}) - space:

$$\omega^2 - |k|^2 = \bar{m}^2 = \text{const}\tag{7}$$

where we call \bar{m} a generalized mass whose relation to the physical mass will be found later.

To find the explicit expressions of p_j and p_0 as functions of the boost parameter ξ we write the derivative $\frac{d\omega}{d\xi}$

$$k_1 = \frac{d\omega}{d\xi} = \frac{\partial \omega}{\partial p_0} \frac{\partial p_0}{\partial \xi} + \sum \frac{\partial \omega}{\partial p_j} \frac{\partial p_j}{\partial \xi}\tag{8}$$

Using the differential equations satisfied by (p_0, p_j) [?] we obtain from (8) the following characteristic equations:

$$\frac{d\omega}{k_1} = \frac{dp_0}{p_1} = - \frac{dp_1}{\frac{\lambda}{2}[p_1^2 - p_2^2 - p_3^2 - (1 - e^{-\lambda p_0})/\lambda^2]} = - \frac{dp_2}{\lambda p_1 p_2} = - \frac{dp_3}{\lambda p_1 p_3}\tag{9}$$

From the first and two last equations we obtain the following two invariants:

$$p_2 e^{\lambda p_0} = \text{const} = k_2, \quad p_3 e^{\lambda p_0} = \text{const} = k_3 \quad (10)$$

Inserting (10) into the second and third equations of (9), we obtain the following equations in complete differentials:

$$dp_0 \lambda \frac{p_1^2}{2} e^{\lambda p_0} + e^{\lambda p_0} p_1 dp_1 + d\left(\frac{p_2^2 e^{\lambda p_0}}{2} + \frac{p_3^2 e^{\lambda p_0}}{2}\right) - \frac{1}{\lambda^2} d[\cosh(\lambda p_0)] = 0 \quad (11)$$

yielding another invariant (Casimir operator of the original algebra) [1]:

$$\cosh(\lambda p_0) - \frac{\lambda^2}{2} |p|^2 e^{\lambda p_0} = \text{const} \quad (12)$$

To find the *const* on the right hand side of (12) we notice that if the momentum $|p| = 0$ then energy p_0 is simply particle's mass. This gives the constant's value:

$$\text{const} = \cosh(\lambda m) \quad (13)$$

Equation (12) can now be identically rewritten

$$\left(\frac{2 \sinh \frac{p_0 \lambda}{2}}{\lambda}\right)^2 - p^2 e^{\lambda p_0} = \left(\frac{2 \sinh \frac{m \lambda}{2}}{\lambda}\right)^2 \quad (14)$$

To find the explicit expressions for ω and k_1 we use eq.(7) to represent k_1 in terms of ω and eq.(12) to represent p_1 in terms of p_0 . Upon substitution of the results into the first and second equations of (9) we arrive at the following elementary quadratures:

$$\int \frac{d\omega}{\sqrt{\omega^2 - D^2}} = \int \frac{dx}{\sqrt{x^2 - 2x \cosh(\lambda m) + 1 - \lambda^2(k_2^2 + k_3^2)}} \quad (15)$$

where

$$D^2 = \bar{m}^2 + k_2^2 + k_3^2 \quad \text{and} \quad x \equiv p_0 e^{\lambda p_0}.$$

Using the fact that k_2 and k_3 are constant we obtain from (15):

$$\begin{aligned} \log(k_1 + \omega) &= \log\left[p_1 e^{\lambda p_0} + \frac{e^{\lambda p_0} - \cosh(\lambda m)}{\lambda}\right] \Rightarrow \\ &\quad k_1 = e^{\lambda p_0}, \\ \omega &= \frac{e^{\lambda p_0} - \cosh(\lambda m)}{\lambda} \end{aligned} \quad (16)$$

This expressions together with eq.(10) completely determine the transformation from p_j - p_0 -space to k_j - ω -space. If we insert (10) and 16) into the dispersion equation (7) and take into account that for $|k| = 0$ $\omega = \bar{m}$ we get the expression for the reduced mass in terms of the mass m ^[1]

$$\bar{m} = \frac{\sinh(\lambda m)}{\lambda}$$

We write all the transformation formulas one more time:

$$\begin{aligned} k_j &= p_j e^{\lambda p_0}, \\ \omega &= \frac{e^{\lambda p_0} - \cosh(\lambda m)}{\lambda}, \\ \bar{m} &= \frac{\sinh(\lambda m)}{\lambda} \end{aligned} \quad (17)$$

We still have to express ω and k_1 in terms of the boost parameter ξ . From eqs.(3)we obtain:

$$\begin{aligned} \omega &= \omega_0 \cosh \xi + k_{10} \sinh \xi \\ k_1 &= k_{10} \cosh \xi + \omega_0 \sinh \xi \end{aligned} \quad (18)$$

where k_{10} and ω_0 are the values of the respective quantities at $\xi = 0$.

Both these expressions together with the transformation rules furnished by (17) give us (after simple algebra) the values of p_0 and p_j for arbitrary initial ($\xi = 0$) conditions, that is for arbitrary values of p_{j0} :

$$p_0 = \frac{1}{\lambda} \log \left\{ \frac{1}{1 - \lambda^2 |p_0|^2} [R + \lambda^2 |p_0|^2 \cosh(\lambda m)] \cosh \xi + \lambda p_{10} [R + \cosh(\lambda m)] \sinh \xi + \cosh \lambda m (1 - \lambda^2 |p_0|^2) \right\}$$

$$p_1 = \frac{(1 - \lambda^2 |p_0|^2) R_1 \sinh \xi + p_{10} [\cosh(\lambda m) + R] \cosh \xi}{[R + \lambda^2 |p_0|^2 \cosh(\lambda m)] \cosh \xi + \lambda p_{10} [R + \cosh(\lambda m)] \sinh \xi + \cosh \lambda m (1 - \lambda^2 |p_0|^2)}$$

$$p_{2,3} = \frac{p_{20,30} [\cosh(\lambda m) + R]}{[R + \lambda^2 |p_0|^2 \cosh(\lambda m)] \cosh \xi + \lambda p_{10} [R + \cosh(\lambda m)] \sinh \xi + \cosh \lambda m (1 - \lambda^2 |p_0|^2)} \quad (19)$$

where

$$R = \sqrt{\sinh^2(\lambda m) + \lambda^2 |p_0|^2}$$

and

$$R_1 = \sqrt{\frac{\sinh^2(\lambda m)}{\lambda^2} + |p_0|^2}$$

For a particle initially at rest $|p_0| = 0$ eqs.(19) are simplified, yielding the result discussed in [3].

For $\xi \rightarrow \infty$ and $\lambda\xi \neq 0$ eqs.(240) become:

$$\begin{aligned}\omega &= \frac{\xi}{\lambda} + O(\xi^{-1}), \\ p_1 &= \frac{(1 - \lambda^2 k^2)R_1 + p_{10}[\cosh(\lambda m) + R]}{R + \lambda^2 |p_0|^2 \cosh(\lambda m) + \lambda p_{10}[\cosh(\lambda m) + R]} + O(e^{-\xi}), \\ p_{2,3} &\rightarrow 0\end{aligned}\quad (20)$$

We consider now the transformation rules for x_0 and x_i introduced in [3]. Our point of departure would be somewhat different from [3]. We begin by directly introducing the Minkowski distance in $x_0 - x_i$ -space

$$ds^2 = dx_0^2 - dx_i^2 \equiv dx_0^2(1 - V_i^2) \quad (21)$$

where

$$V_i = \frac{dx_i}{dx_0} \quad (22)$$

The action according to the conventional prescription is

$$S = -a \int ds = -a \int dx_0 \sqrt{1 - V_i^2} \quad (23)$$

The respective lagrangian looks exactly as the standard special-relativistic lagrangian:

$$\mathcal{L} = -a\sqrt{1 - V_i^2}.$$

The momentum p_i is then

$$p_i = \frac{\partial \mathcal{L}}{\partial V_i} = a \frac{V_i}{\sqrt{1 - V_i^2}} \quad (24)$$

If we substitute p_i from (19), where for simplicity sake we take $|p_0| = 0$, we get the following equation:

$$p_i = \frac{\bar{m} \sinh \xi}{\lambda \bar{m} \cosh \xi + \sqrt{1 + \bar{m}^2 \lambda^2}} = a \frac{V}{\sqrt{1 - V^2}} \quad (25)$$

Now we make an *ansatz*

$$p = \bar{m} \frac{V}{\sqrt{1 - V^2}} \quad (26)$$

Comparison of (25) and (26) yields $a = \bar{m}$, and as a result the value of V [3].

$$V = \frac{\sinh \xi}{\cosh \xi \sqrt{1 + \bar{m}^2 \lambda^2} + \sqrt{1 + \bar{m}^2 \lambda^2}} \quad (27)$$

The most interesting part of this derivation is that the compatibility of (21), (22), and (27), requires

$$\frac{dx_0}{d\xi} \neq x; \quad \frac{dx}{dx_0} \neq x_0$$

The way out of this situation is to introduce a modified boost parameter $z = z(\xi)$ in $x - x_0$ -space, different from the "bare" boost parameter ξ in $p - p_0$ -space:

$$x = x' \cosh z + x'_0 \sinh z; \quad x_0 = x'_0 \cosh z + x' \sinh z;$$

where prime denote an inertial system K' moving uniformly with respect to another inertial system K .

If initially, a particle was at rest, then $x/x_0 = V = \tanh z$ and comparing this expression with eq.(21) we find the value of the boost parameter z [3]:

$$z(\xi) = 2 \tanh^{-1} [e^{m\lambda} \tanh(\frac{\xi}{2})]$$

Impossibility to simultaneously introduce standard Minkowski metric in both $x - x_0$ and $p - p_0$ -spaces is due to the fact that in the respective algebra x and x_0 do not commute. Moreover, the $k - \omega$ -space has the Minkowski structure, albeit at the expense of abandoning the standard relativistic transition

$$\omega \Leftrightarrow p_0; \quad k \Leftrightarrow p.$$

This problem was discussed in detail in [4]. In the latter the obtained dispersion equation differed from the one used in the later studies (e.g., [1], [3]) and discussed here.

However what not mentioned is an absence of a *unique* definition of the time derivative in the presence of the minimum attainable length λ , even if such a definition is restricted by a choice of a suitable algebra. This non-uniqueness is an important factor in the emergence of different dispersion equations in $p - p_0$ -space. Therefore it would look like an unpleasant fact.

However, in a transition to the correct dispersion relation $\omega - k$ -space this difference (within the framework of the chosen algebra, κ -Poincare) does not play any role, as will be shown below.

In general, one can define such a derivative as follows:

$$(\frac{\partial f}{\partial t})_\lambda \equiv \frac{f(t + \alpha\lambda) - f(t - \beta\lambda)}{\gamma\lambda} = \frac{1}{\gamma\lambda}(e^{\alpha\frac{\partial}{\partial t}} - e^{-\beta\frac{\partial}{\partial t}})f \quad (28)$$

with the following relation between parameters α, β , and γ :

$$\alpha + \beta = \gamma.$$

For the conventional definition of the derivative the choice of the parameters is not important, and all different definitions are equivalent.

In work [4] the parameters were:

$$\alpha = 0, \beta = i, \gamma = i$$

In the subsequent studies (e.g., [1], [3]) this choice was abandoned in favor of

$$\alpha = \beta = \frac{1}{2}, \gamma = 1.$$

The respective transition $\omega \Leftrightarrow p_0$, $k \Leftrightarrow$ is as follows:

$$\alpha = 0, \beta = i, \gamma = i; \quad \omega^- = \frac{1 - e^{-\lambda p_0}}{\lambda}; \quad k = p e^{-\lambda p_0}$$

$$\alpha = \beta = \frac{1}{2}, \gamma = 1; \quad \omega^+ = \frac{e^{\lambda p_0} - 1}{\lambda}; \quad k = p e^{\lambda p_0}$$

Still the dispersion equations in $\omega - k$ -space coincide for both cases, yielding (7)

$$\omega^2 - k^2 = \bar{m}^2.$$

Note the ubiquitousness of the factor $e^{\pm p_0 \lambda}$, corresponding to the shifting operator.

3 Conclusion

We have provided an alternative (rather elementary) derivation of the results of the work [1] using the standard Lorentz group for the representation of the algebra governing the space $\omega - k$. It has turned out that not only the derivation is simpler, but more important it automatically yields both the

dispersion equation in $p - p_0$ -space and $\omega - k$ -space, where the latter is consistent with the velocity of a massless particle.

Interestingly enough, the non-uniqueness of the definition of the (time) derivative in the environment with the minimum length scale does not affect the final result, the dispersion equation in $\omega - k$ -space, although the dispersion equations in $p - p_0$ -space are different for different choices of the time derivative.

It would be interesting to investigate also a situation where one would take into account not only the explicit inclusion of the minimum length scale into the commutation relations but also of the minimum time scale.

References

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